

ON THE DISTRIBUTION OF αp MODULO ONE FOR PRIMES p OF A SPECIAL FORM

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ABSTRACT. A classical problem in analytic number theory is to study the distribution of αp modulo 1, where α is irrational and p runs over the set of primes. We consider the subsequence generated by the primes p such that $p + 2$ is an almost-prime (the existence of infinitely many such p is another topical result in prime number theory) and prove that its distribution has a similar property.

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1. INTRODUCTION AND STATEMENTS OF THE RESULT

The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. This hypothesis is still unproved but there are many approximation to it established. One of the most interesting of them is due to Chen [1]. In 1973 he proved that there are infinitely many primes p for which $p + 2 = P_2$. (As usual P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

Suppose that we have a problem including primes and let $r \geq 2$ be an integer. Having in mind Chen's result we may consider this problem with primes p , such that $p + 2 = P_r$. We will give several examples.

In 1937, Vinogradov [16] proved that every sufficiently large odd n can be represented in the form

$$(1) \quad p_1 + p_2 + p_3 = n,$$

where p_1, p_2, p_3 are primes. In 2000 Peneva [12] and Tolev [13] considered (1) with primes of the form specified above. It was established in [13] that if n is sufficiently large and $n \equiv 3 \pmod{6}$ then (1) has a solution in primes p_1, p_2, p_3 such that

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.$$

Further, in 1938 Hua [8] proved that every sufficiently large $n \equiv 5 \pmod{24}$ can be represented as

$$(2) \quad p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = n,$$

where p_1, \dots, p_5 are primes. In 2000 Tolev [15] proved that every sufficiently large $n \equiv 5 \pmod{24}$ can be represented in the form (2) with

primes p_1, \dots, p_5 such that

$$p_1 + 2 = P_2, p_2 + 2 = P'_2, p_3 + 2 = P_5, p_4 + 2 = P'_5, p_5 + 2 = P_7.$$

Finally, in 2004 Green and Tao [3] proved their celebrated theorem stating that for every natural $k \geq 3$ there are infinitely many arithmetical progression of k different primes. Later they established (see [4]) that there exist infinitely many arithmetical progression of three different primes p , such that $p + 2 = P_2$. (A weaker result of this type was previously obtained by Tolev [14]). In the paper [4] Green and Tao state that using the same method their result can be extended for progression of k terms, where k is arbitrary large.

In the present paper we consider another popular problem with primes and study it with primes of the form specified above.

Let α be irrational real number, β be real and let $||x|| = \min_{n \in \mathbb{Z}} |x - n|$.

In 1947 Vinogradov [17] proved that if $0 < \theta < 1/5$ then there are infinitely many primes p such that

$$||\alpha p + \beta|| < p^{-\theta}.$$

Latter the upper bound for θ was improved and the strongest published result is due to Heath-Brown and Jia [7] with $\theta < 16/49$. We shall prove the following

Theorem 1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \in \mathbb{R}$ and let $0 < \theta \leq 1/100$. Then there are infinitely many primes p satisfying $p + 2 = P_4$ and such that*

$$(3) \quad ||\alpha p + \beta|| < p^{-\theta}.$$

Other versions of this theorem are also possible, but our intention is to present here a result with r as small as possible and for this r to find some (not necessarily the biggest possible) θ .

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2. NOTATION

Let N be a sufficiently large real number and $\delta, \eta, \theta, \rho, \kappa$ be real constants satisfying

$$(4) \quad 0 < \theta < \eta < \frac{\delta}{2} < \frac{1}{4}, \quad \eta < \rho < \delta, \quad 0 < \kappa, \quad 0 < \theta \leq \frac{1}{100}.$$

We shall specify δ, η, ρ and κ latter. We put

$$(5) \quad \begin{aligned} z &= N^\eta, \quad y = N^\rho, \quad D = N^\delta, \\ \Delta &= \Delta(N) = N^{-\theta}, \quad H = \Delta^{-1} \log^2 N. \end{aligned}$$

By p and q we always denote primes. As usual $\Omega(n)$, $\varphi(n)$, $\mu(n)$, $\Lambda(n)$ denote respectively the numbers of prime factors of n counted with the

multiplicity, Euler's function, Möbius' function and Mangoldt's function. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \dots m_k = n$ in natural numbers m_1, \dots, m_k and $\tau_2(n) = \tau(n)$. Let (m_1, \dots, m_k) and $[m_1, \dots, m_k]$ be the greatest common divisor and the least common multiple of m_1, \dots, m_k respectively. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of y , $||y||$ – the distance from y to the nearest integer, $e(y) = e^{2\pi i y}$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$. The letter ε denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^\varepsilon \log x \ll x^\varepsilon$.

3. PROOF OF THE THEOREM

We take a periodic with period 1 function such that

$$(6) \quad \begin{aligned} 0 < \chi(t) < 1 & \quad \text{if} \quad -\Delta < t < \Delta; \\ \chi(t) = 0 & \quad \text{if} \quad \Delta \leq t \leq 1 - \Delta, \end{aligned}$$

and which has a Fourier series

$$(7) \quad \chi(t) = \Delta + \sum_{|k| > 0} g(k) e(kt),$$

with coefficients satisfying

$$(8) \quad \begin{aligned} g(0) &= \Delta, \\ g(k) &\ll \Delta \quad \text{for all } k, \\ \sum_{|k| > H} |g(k)| &\ll N^{-1}. \end{aligned}$$

The existence of such a function is a consequence of a well known lemma of Vinogradov (see [11], ch. 1, §2).

Consider the sum

$$(9) \quad \Gamma = \Gamma(N) = \sum_{\substack{N/2 < p \leq N \\ (p+2, P(z))=1}} \chi(\alpha p + \beta) T_p \log p,$$

where

$$(10) \quad P(z) = \prod_{2 < p \leq z} p$$

and

$$(11) \quad T_p = 1 - \kappa \sum_{\substack{z < q \leq y \\ q|p+2}} \left(1 - \frac{\log q}{\log y} \right).$$

Obviously

$$(12) \quad \Gamma(N) \leq \Gamma_1$$

where Γ_1 is the sum of the terms of $\Gamma(N)$ for which $T_p > 0$. Denote by Γ_2 the sum of the term of Γ_1 for which $\mu^2(p+2) = 0$. It is clear that

$$(13) \quad 0 \leq \Gamma_2 \ll \sum_{z \leq q} \sum_{\substack{n \leq N \\ n+2 \equiv 0 \pmod{q^2}}} \log n \ll \log N \sum_{z \leq q \leq \sqrt{N+2}} \left(\frac{N}{q^2} + 1 \right) \\ \ll \frac{N^{1+\varepsilon}}{z} + N^{\frac{1}{2}+\varepsilon} \ll N^{1-\eta+\varepsilon}.$$

We also remove from Γ_1 the term (if such exist) for which $N-2 < p \leq N$ and the resulting error is $O(\log N)$. Therefore

$$(14) \quad \Gamma \leq \Gamma_3 + O(N^{1-\eta+\varepsilon}),$$

where

$$\Gamma_3 = \sum \chi(\alpha p + \beta) T_p \log p$$

and where the summation is taken over the primes p , satisfying

$$(15) \quad N/2 < p \leq N-2,$$

$$(16) \quad T_p > 0, \quad \mu^2(p+2) = 1, \quad (p+2, P(z)) = 1.$$

Assume that

$$(17) \quad \Gamma(N) \gg \frac{\Delta N}{\log N}.$$

Then from (4), (5) and (14) we get $\Gamma_3 > 0$. Hence there exist a prime p satisfying (15), (16) and such that

$$(18) \quad \chi(\alpha p + \beta) > 0.$$

From (5), (6), (15) and (18) it follows that this prime satisfies (3).

On the other hand, from the properties of the weights T_p (see [5], ch. 9) it follows that if p satisfies (15), (16) then

$$\Omega(p+2) \leq \frac{1}{\kappa} + \frac{1}{\rho}.$$

We see that to prove our theorem it is enough to determine the constants $\delta, \eta, \theta, \rho, \kappa$ in such a way that:

I: There exist a sequence $\{N_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \rightarrow \infty} N_j = \infty, \quad \Gamma(N_j) \gg \frac{\Delta(N_j) N_j}{\log N_j}, \quad j = 1, 2, 3, \dots$$

II: We have

$$(19) \quad \frac{1}{\kappa} + \frac{1}{\rho} < 5$$

Using (9) and (11) we write Γ as

$$(20) \quad \Gamma = \Phi - \kappa G,$$

where

$$(21) \quad \Phi = \sum_{\substack{N/2 < p \leq N \\ (p+2, P(z))=1}} \chi(\alpha p + \beta) \log p$$

and

$$(22) \quad G = \sum_{\substack{N/2 < p \leq N \\ (p+2, P(z))=1}} \chi(\alpha p + \beta) \log p \sum_{\substack{z < q \leq y \\ q|p+2}} \left(1 - \frac{\log q}{\log y}\right).$$

We shall estimate Φ from below and G from above.

Consider the sum Φ . We apply a lower bound linear sieve. We take the lower Rosser weights $\lambda^-(d)$ of order D . For the definition and their properties we refer the reader to [9], [10]. In particular we shall use that the Rosser weights are real numbers such that

$$(23) \quad |\lambda^-(d)| \leq 1, \quad \lambda^-(d) = 0 \quad \text{if } d > D \quad \text{or} \quad \mu^2(d) = 0,$$

$$(24) \quad \sum_{d|A} \lambda^-(d) \leq \begin{cases} 1, & \text{if } A = 1, \\ 0, & \text{if } A \in \mathbb{N}, A > 1. \end{cases}$$

We shall also use that if

$$(25) \quad s = \frac{\log D}{\log z} = \frac{\delta}{\eta} \quad \text{and} \quad 2 < s < 4$$

then

$$(26) \quad \sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} \geq \Pi(z) \left(\frac{2e^\gamma \log(s-1)}{s} + O\left((\log N)^{-1/3}\right) \right),$$

where

$$(27) \quad \Pi(z) = \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right).$$

From this place onwards we assume that

$$(28) \quad 2 < \frac{\delta}{\eta} < 4,$$

so the inequality (26) holds. Using (21), (24) we get

$$(29) \quad \begin{aligned} \Phi &\geq \Phi_1 = \sum_{N/2 < p \leq N} \chi(\alpha p + \beta) \log p \sum_{d|(p+2, P(z))} \lambda^-(d) \\ &= \sum_{d|P(z)} \lambda^-(d) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{d}}} \chi(\alpha p + \beta) \log p. \end{aligned}$$

Form (7), (8) we find that

$$(30) \quad \Phi_1 = \Delta(\Phi_2 + \Phi_3) + O(1),$$

where

$$\begin{aligned} \Phi_2 &= \sum_{d|P(z)} \lambda^-(d) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{d}}} \log p, \\ (31) \quad \Phi_3 &= \sum_{d|P(z)} \lambda^-(d) \sum_{0 < |k| \leq H} c(k) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{d}}} e(\alpha p k) \log p, \end{aligned}$$

$$(32) \quad c(k) = \Delta^{-1} g(k) e(\beta k) \ll 1.$$

Consider Φ_2 . From Bombieri-Vinogradov's theorem (see [2], ch. 24), (4), (5), (23) we see that

$$(33) \quad \Phi_2 = \frac{N}{2} \sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} + O\left(\frac{N}{(\log N)^2}\right).$$

It is well known that the product defined by (27) satisfies

$$(34) \quad \Pi(z) \asymp \frac{1}{\log z}.$$

Therefore from (4), (5), (26), (33), (34) we find that

$$(35) \quad \Phi_2 \geq e^\gamma N \Pi(z) \frac{\log(s-1)}{s} + O\left(\frac{N}{(\log N)^{4/3}}\right),$$

where s is specified by (25). We shall study the sum Φ_3 later.

Consider now the sum G , defined by (22). We write it in the form

$$(36) \quad G = \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{q} \\ (p+2, P(z))=1}} \chi(\alpha p + \beta) \log p$$

and then apply an upper bound linear sieve. Let $\lambda_q(d)$ be the upper Rosser weights of order $\frac{D}{q}$. We know that

$$(37) \quad |\lambda_q(d)| \leq 1, \quad \lambda_q(d) = 0 \quad \text{if } d > \frac{D}{q} \quad \text{or } \mu^2(d) = 0,$$

$$(38) \quad \sum_{d|A} \lambda_q(d) \geq \begin{cases} 1, & \text{if } A = 1, \\ 0, & \text{if } A \in \mathbb{N}, A > 1. \end{cases}$$

We shall also use that if

$$(39) \quad s_1 = \frac{\log(D/q)}{\log z} \quad \text{and} \quad 1 < s_1 < 3$$

then

$$(40) \quad \sum_{d|P(z)} \frac{\lambda_q(d)}{\varphi(d)} \leq \Pi(z) \left(\frac{2e^\gamma}{s_1} + O((\log N)^{-1/3}) \right)$$

From this place onwards we assume that

$$(41) \quad \eta + \rho < \delta.$$

Then using also (28) we see that the condition (39) holds, consequently (40) is true. From (36) – (38) we find

$$\begin{aligned}
 (42) \quad G &\leq G_1 = \\
 &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{q}}} \chi(\alpha p + \beta) \log p \sum_{d|(p+2, P(z))} \lambda_q(d) \\
 &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \lambda_q(d) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{qd}}} \chi(\alpha p + \beta) \log p \\
 &= \sum_{m \leq D} \gamma(m) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{m}}} \chi(\alpha p + \beta) \log p,
 \end{aligned}$$

where

$$(43) \quad \gamma(m) = \sum_{\substack{z < q < y \\ d|P(z) \\ qd=m}} \left(1 - \frac{\log q}{\log y}\right) \lambda_q(d).$$

Using (10), (37) and (43) we easily see that

$$(44) \quad |\gamma(m)| \leq 1.$$

From (7), (8) and (42) we find

$$(45) \quad G_1 = \Delta(G_2 + G_3) + O(1),$$

where

$$\begin{aligned}
 (46) \quad G_2 &= \sum_{m \leq D} \gamma(m) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{m}}} \log p, \\
 G_3 &= \sum_{m \leq D} \gamma(m) \sum_{0 < |k| \leq H} c(k) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{m}}} e(\alpha p k) \log p,
 \end{aligned}$$

and where $c(k)$ satisfies (32).

We apply again Bombieri-Vinogradov's theorem and (4), (5), (44) to find that

$$(47) \quad G_2 = \frac{N}{2} \sum_{m \leq D} \frac{\gamma(m)}{\varphi(m)} + O\left(\frac{N}{(\log N)^2}\right).$$

Using (40), (43) we obtain

$$\begin{aligned}
 (48) \quad \sum_{m \leq D} \frac{\gamma(m)}{\varphi(m)} &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \frac{\lambda_q(d)}{\varphi(qd)} \\
 &= \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \sum_{d|P(z)} \frac{\lambda_q(d)}{\varphi(d)} \\
 &\leq \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \\
 &\quad \times \Pi(z) \left(2e^\gamma \left(\frac{\log(D/q)}{\log z}\right)^{-1} + O\left((\log N)^{-1/3}\right)\right).
 \end{aligned}$$

Therefore from (5), (34), (47), (48) we get

(49)

$$G_2 \leq e^\gamma N \Pi(z) \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1} + O\left(\frac{N}{(\log N)^{4/3}}\right).$$

Now we are in a position to find a lower bound for the sum Γ . From (20), (29), (30), (35), (42), (45), (49) it follows that

$$(50) \quad \Gamma \geq e^\gamma \Delta N \Pi(z) \Sigma + O\left(\frac{\Delta N}{(\log N)^{4/3}}\right) + O\left(\Delta |\Phi_3 - \kappa G_3|\right),$$

where

$$(51) \quad \Sigma = \frac{\log(s-1)}{s} - \kappa \sum_{z < q < y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1}$$

and where s is specified by (25). Using partial summation and the prime number theorem it is easy to prove that

$$(52) \quad \Sigma = \Sigma_0 + O\left(\frac{1}{\log N}\right),$$

where

$$(53) \quad \Sigma_0 = \frac{\log(s-1)}{s} - \kappa \eta \int_{\eta}^{\rho} \left(\frac{1}{u} - \frac{1}{\rho}\right) \frac{1}{\delta - u} du.$$

Therefore, using (5), (34), (50) we get

$$\Gamma \geq e^\gamma N \Delta \Pi(z) \Sigma_0 + O\left(\frac{\Delta N}{(\log N)^{4/3}}\right) + O\left(\Delta |\Phi_3 - \kappa G_3|\right).$$

Now we shall see that if N is a term of a suitable sequence tending to infinity then the last error term in the formula above can be omitted. The following lemma holds:

Lemma 1. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and*

$$(54) \quad \delta + \theta < \frac{1}{3}.$$

Let $\xi(d)$, $c(k)$ be complex numbers defined for $d \leq D$, $0 < |k| \leq H$, where D and H are specified by (5), and let

$$(55) \quad \xi(d) \ll 1, \quad c(k) \ll 1.$$

Then there exist a sequence $\{N_j\}_{j=1}^\infty$, $\lim_{j \rightarrow \infty} N_j = \infty$, such that if

$$(56) \quad S(N) = \sum_{d \leq D} \xi(d) \sum_{1 \leq |k| \leq H} c(k) \sum_{\substack{N/2 < p \leq N \\ p+2 \equiv 0 \pmod{d}}} e(\alpha p k) \log p$$

then we have

$$S(N_j) \ll \frac{N_j}{\log^2 N_j}, \quad j = 1, 2, 3, \dots$$

We will present the proof of this Lemma in the next section.

From (31), (46) we see that the quantity $\Phi_3 - \kappa G_3$ can be written as a sum of type (56) with $\xi(d) = \lambda^*(d) - \kappa \gamma(d)$, where $\lambda^*(d) = \lambda^-(d)$ if $d|P(z)$ and $\lambda^*(d) = 0$ otherwise. Using our Lemma we see that there exist a sequence tending to infinity such that if N is its term then

$$(57) \quad \Gamma \geq e^\gamma \Delta N \Pi(z) \Sigma_0 + O\left(\frac{\Delta N}{(\log N)^{4/3}}\right).$$

We put

$$\rho = 0.23, \quad \delta = 0.315, \quad \eta = 0.08, \quad \kappa = 1.58.$$

Then it is easy to verify that the conditions (4), (19), (28), (41), (54) are fulfilled and also

$$\Sigma_0 > 0.$$

From the last inequality and (5), (34), (57) it follows that there exist a constant $c > 0$ such that for any N from our sequence we have

$$\Gamma \geq c \frac{\Delta N}{\log N} > 0.$$

This completes the proof of the theorem.

4. PROOF OF LEMMA

Since α is irrational we see, using Dirichlet's theorem, that there are infinitely many integers A and natural numbers Q such that

$$\left| \alpha - \frac{A}{Q} \right| < \frac{1}{Q^2}.$$

For any such Q we choose N in a suitable way (see (73)) and in this way define our sequence $\{N_j\}_{j=1}^\infty$.

It is clear that

$$(58) \quad S(N) = W + O(HN^{\frac{1}{2}+\varepsilon}),$$

where

$$W = \sum_{N/2 < n \leq N} \Lambda(n) \sum_{1 \leq |k| \leq H} c(k)e(\alpha nk) \sum_{\substack{d \leq D \\ d|n+2 \\ 2 \nmid d}} \xi(d).$$

Using Heath-Brown's identity [6] with parameters

$$(59) \quad P = N/2, P_1 = N, u = 2^{-7}N^{\frac{\delta}{2}}, v = 2^7N^{\frac{1}{3}}, w = N^{\frac{1}{2}-\frac{\delta}{4}}.$$

we decompose the sum W as a linear combination of $O(\log^6 N)$ sums of first and second type. The sums of the first type are

$$W_1 = \sum_{M < m \leq M_1} a_m \sum_{\substack{L < l \leq L_1 \\ N/2 < ml \leq N}} \sum_{0 < |k| \leq H} c(k)e(\alpha mlk) \sum_{\substack{d \leq D \\ d|ml+2 \\ 2 \nmid d}} \xi(d)$$

and

$$W'_1 = \sum_{M < m \leq M_1} a_m \sum_{\substack{L < l \leq L_1 \\ N/2 < ml \leq N}} \log l \sum_{0 < |k| \leq H} c(k)e(\alpha mlk) \sum_{\substack{d \leq D \\ d|ml+2 \\ 2 \nmid d}} \xi(d),$$

where

$$(60) \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad ML \asymp N, \quad L \geq w, \quad a_m \ll N^\varepsilon.$$

The sums of the second type are

$$W_2 = \sum_{M < m \leq M_1} a_m \sum_{\substack{L < l \leq L_1 \\ N/2 < ml \leq N}} b_l \sum_{0 < |k| \leq H} c(k)e(\alpha mlk) \sum_{\substack{d \leq D \\ d|ml+2 \\ 2 \nmid d}} \xi(d),$$

where

$$(61) \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad ML \asymp N, \quad u \leq L \leq v, \quad a_m, b_l \ll N^\varepsilon.$$

First we estimate the sums of second type. We have

$$W_2 \ll N^\varepsilon \sum_{M < m \leq M_1} \left| \sum_{\substack{L < l \leq L_1 \\ N/2 < ml \leq N}} b_l \sum_{0 < |k| \leq H} c(k)e(\alpha mlk) \sum_{\substack{d \leq D \\ d|ml+2 \\ 2 \nmid d}} \xi(d) \right|.$$

Applying the Cauchy inequality and (55), (61) we get

$$\begin{aligned} |W_2|^2 &\ll N^\varepsilon M \sum_{M < m \leq M_1} \left| \sum_{\substack{L < l \leq L_1 \\ N/2 < ml \leq N}} b_l \sum_{0 < |k| \leq H} c(k)e(\alpha mlk) \sum_{\substack{d \leq D \\ d|ml+2 \\ 2 \nmid d}} \xi(d) \right|^2 \\ &\ll N^\varepsilon M \sum_{\substack{d_1, d_2 \leq D \\ 2 \nmid d_1 d_2}} \sum_{0 < k_1, k_2 \leq H} \sum_{L < l_1, l_2 \leq L_1} |V|, \end{aligned}$$

where

$$V = \sum_{\substack{M' < m \leq M'_1 \\ l_i m + 2 \equiv 0(d_i), i=1,2}} e(\alpha m(k_1 l_1 - k_2 l_2)),$$

$$M' = \max \left\{ \frac{N}{2l_1}, \frac{N}{2l_2}, M \right\}, \quad M'_1 = \min \left\{ \frac{N}{l_1}, \frac{N}{l_2}, M_1 \right\}.$$

If the system of congruences

$$(62) \quad \begin{cases} l_1 m + 2 \equiv 0(d_1) \\ l_2 m + 2 \equiv 0(d_2). \end{cases}$$

has no solution then $V = 0$. Assume that the system (62) has a solution. Then there exist an integer $f = f(l_1, l_2, d_1, d_2)$ such that (62) is equivalent to $m \equiv f([d_1, d_2])$ and therefore

$$\begin{aligned} V &= \sum_{\substack{M' < m \leq M'_1 \\ m \equiv f([d_1, d_2])}} e(\alpha m(k_1 l_1 - k_2 l_2)) \\ &= e(\alpha f(k_1 l_1 - k_2 l_2)) \sum_{\substack{\frac{M' - f}{[d_1, d_2]} < s \leq \frac{M'_1 - f}{[d_1, d_2]}} e(\alpha s[d_1, d_2](k_1 l_1 - k_2 l_2)). \end{aligned}$$

From (5), (54), (59), (61) it follows that

$$(63) \quad M \gg \frac{N}{v} \gg D^2.$$

Applying Lemma 4 from [11], ch. 6, §2, we get

$$(64) \quad V \ll \begin{cases} \frac{M}{[d_1, d_2]}, & \text{if } k_1 l_1 = k_2 l_2, \\ \min \left\{ \frac{M}{[d_1, d_2]}, \frac{1}{\|\alpha(k_1 l_1 - k_2 l_2)[d_1, d_2]\|} \right\}, & \text{if } k_1 l_1 \neq k_2 l_2. \end{cases}$$

Therefore

$$(65) \quad |W_2|^2 \ll N^\varepsilon M \left(M V_1 + V_2 \right),$$

where

$$\begin{aligned} V_1 &= \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \sum_{0 < k_1, k_2 \leq H} \sum_{\substack{L < l_1, l_2 \leq L_1 \\ k_1 l_1 = k_2 l_2}} 1, \\ V_2 &= \sum_{d_1, d_2 \leq D} \sum_{0 < k_1, k_2 \leq H} \sum_{\substack{L < l_1, l_2 \leq L_1 \\ k_1 l_1 \neq k_2 l_2}} \min \left\{ \frac{M}{[d_1, d_2]}, \frac{1}{\|\alpha(k_1 l_1 - k_2 l_2)[d_1, d_2]\|} \right\}. \end{aligned}$$

It is clear that

$$(66) \quad V_1 \ll \sum_{h \leq D^2} \frac{1}{h} \sum_{[d_1, d_2] = h} 1 \sum_{n \leq 2HL} \tau^2(n) \ll N^\varepsilon HL \sum_{h \leq D^2} \frac{\tau^2(h)}{h} \ll N^\varepsilon HL.$$

Consider V_2 we have

$$\begin{aligned} V_2 &\ll \sum_{h \leq D^2} \tau^2(h) \sum_{0 < |r| \leq 2HL} \min \left\{ \frac{M}{h}, \frac{1}{||\alpha r h||} \right\} \sum_{\substack{0 < n_1, n_2 \leq 2HL \\ n_1 - n_2 = r}} \tau(n_1) \tau(n_2) \\ &\ll N^\varepsilon HL \sum_{h \leq D^2} \sum_{0 < r \leq 2HL} \min \left\{ \frac{M}{h}, \frac{1}{||\alpha r h||} \right\} \\ &\ll N^\varepsilon HL \sum_{m \leq 2D^2 HL} \min \left\{ \frac{HLM}{m}, \frac{1}{||\alpha m||} \right\}. \end{aligned}$$

Since $M \gg D^2$ (see (63)) we can apply Lemma 2.2 from [18], ch. 2, §2.1 and get

$$(67) \quad V_2 \ll N^\varepsilon \left(\frac{H^2 L^2 M}{Q} + D^2 H^2 L^2 + HLQ \right).$$

From (61), (65) – (67) we obtain

$$\begin{aligned} |W_2|^2 &\ll N^\varepsilon \left(HLM^2 + \frac{H^2 L^2 M^2}{Q} + D^2 H^2 L^2 M + HLMQ \right) \\ &\ll N^\varepsilon \left(\frac{HN^2}{u} + \frac{H^2 N^2}{Q} + D^2 H^2 Nv + HNQ \right). \end{aligned}$$

Hence

$$(68) \quad W_2 \ll N^\varepsilon \left(\frac{H^{\frac{1}{2}} N}{u^{\frac{1}{2}}} + \frac{HN}{Q^{\frac{1}{2}}} + DHN^{\frac{1}{2}} v^{\frac{1}{2}} + H^{\frac{1}{2}} N^{\frac{1}{2}} Q^{\frac{1}{2}} \right).$$

Now we shall estimate the sums of the first type. Using (55), (60) we get

$$(69) \quad W_1 \ll N^\varepsilon \sum_{\substack{d \leq D \\ 2 \nmid d}} \sum_{0 < k \leq H} \sum_{M < m \leq M_1} |U|,$$

where

$$(70) \quad U = \sum_{\substack{L' < l \leq L'_1 \\ ml + 2 \equiv 0(d)}} e(\alpha kml),$$

$$L' = \max \left\{ L, \frac{N}{2m} \right\}, \quad L'_1 = \min \left\{ L_1, \frac{N}{m} \right\}.$$

If $(m, d) > 1$ then the sum U is empty. Suppose now that $(m, d) = 1$. Then the congruence $ml + 2 \equiv 0(d)$ is equivalent to $l \equiv l_0(d)$ for some

integer $l_0 = l_0(m, d)$. Hence we may write U in the form

$$U = e(\alpha k m l_0) \sum_{\frac{L' - l_0}{d} < s \leq \frac{L'_1 + l_0}{d}} e(\alpha k m s d).$$

Using Lemma 4 from [11], ch. 6, §2 we get

$$U \ll \min \left\{ \frac{N}{md}, \frac{1}{\|\alpha k m d\|} \right\},$$

consequently

$$\begin{aligned} W_1 &\ll N^\varepsilon \sum_{d \leq D} \sum_{k \leq H} \sum_{M < m \leq M_1} \min \left\{ \frac{N}{md}, \frac{1}{\|\alpha k m d\|} \right\} \\ &\ll N^\varepsilon \sum_{n \leq 2MD} \sum_{k \leq H} \min \left\{ \frac{N}{n}, \frac{1}{\|\alpha k n\|} \right\} \\ &\ll N^\varepsilon \sum_{n \leq 2MD} \sum_{k \leq H} \min \left\{ \frac{HN}{kn}, \frac{1}{\|\alpha k n\|} \right\} \\ &\ll N^\varepsilon \sum_{s \leq 2MDH} \min \left\{ \frac{NH}{s}, \frac{1}{\|\alpha s\|} \right\}. \end{aligned}$$

Using (54), (59), (60) we see that we may apply again Lemma 2.2, [18], ch. 2, §2.1 and we get

$$(71) \quad W_1 \ll N^\varepsilon \left(\frac{HN}{Q} + \frac{DHN}{w} + Q \right).$$

We consider the sum W'_1 in the same manner and we find

$$(72) \quad W'_1 \ll N^\varepsilon \left(\frac{HN}{Q} + \frac{DHN}{w} + Q \right).$$

From (68), (71) and (72) we obtain

$$W \ll N^\varepsilon \left(\frac{H^{\frac{1}{2}} N}{u^{\frac{1}{2}}} + \frac{HN}{Q^{\frac{1}{2}}} + DHN^{\frac{1}{2}} v^{\frac{1}{2}} + H^{\frac{1}{2}} N^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{DHN}{w} + Q \right).$$

We choose

$$(73) \quad N = Q^{\frac{2}{1+\theta}}$$

and having in mind (4), (54), (59) we obtain

$$W \ll N^{1+\frac{\theta}{2}-\frac{\delta}{4}} + N^{\frac{3(1+\theta)}{4}} + N^{\frac{2}{3}+\delta+\theta} \ll N^{1-\varpi}$$

for some small constant $\varpi > 0$. This proves our Lemma.

REFERENCES

- [1] Chen J. R. , *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica, 16, (1973), 157-176.
- [2] Davenport H., *Multiplicative number theory* (revised by H. Montgomery), Third ed., Springer, 2000.
- [3] Green B., Tao T. *The Primes Contain Arbitrarily Long Arithmetic Progressions*, Annals of Math., to appear.
- [4] Green B., Tao T., *Restriction theory of the Selberg sieve, with applications*, J. Theor. Nombres Bordeaux, 18, (2006), 147–182.
- [5] Halberstam H., Richert H.-E., *Sieve Methods*, Academic Press, London, 1974.
- [6] Heath-Brown D. R., *Prime numbers in short intervals and a generalized Vaughan identity*, Canad. J. Math. 34, (1982), 1365-1377.
- [7] Heath-Brown D. R., Jia C. *The distribution of αp modulo one*, Proc. London Math. Soc., 84, (2002), 396-414.
- [8] Hua L. K., *Some results in the additive prime number theory*, Quart. J. Math. Oxford, 9, (1938), 68-80.
- [9] Iwaniec H., *A new form of the error term in the linear sieve*, Acta Arith., 37, (1980), 307–320.
- [10] Iwaniec H., *Rosser's sieve*, Acta Arith., 36, (1980), 171-202.
- [11] Karatsuba A. A., *Basic analytic number theory*, Nauka, 1983 (in Russian).
- [12] Peneva P., *On the ternary Goldbach problem with primes p such that $p+2$ are almost-prime*, Acta Math. Hungar., 86, (2000), 305–318.
- [13] Tolev D., *Representations of large integers as sums of two primes of special type*, in “Algebraic Number Theory and Diophantine Analysis”, Walter de Gruyter, 2000, 485–495.
- [14] Tolev D., *Arithmetic progressions of prime-almost-prime twins*, Acta Arith., 88, (1999), 67-98.
- [15] Tolev D., *Additive problems with prime numbers of special type*, Acta Arith., 96, (2000), 53-88, Corr. Acta Arith. 105, 2002, 205.
- [16] Vinogradov I. M., *Representation of an odd number as the sum of three primes*, Dokl. Akad. Nauk. SSSR, 15, (1937), 291-294, (in Russian).
- [17] Vinogradov I. M., *The method of trigonometrical sums in the theory of numbers*, Trud. Math. Inst. Steklov, 23, (1947), 1-109, (in Russian).
- [18] Vaughan R. C., *The Hardy-Littlewood Method*, Cambridge Univ. Press, Second ed. 1997.

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